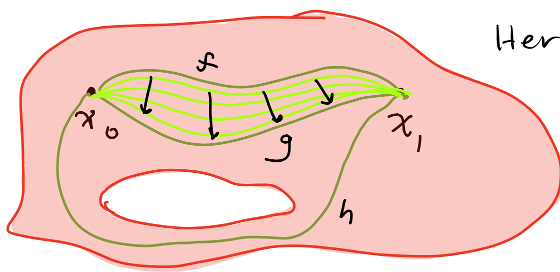
Paths

Let X be a topological space. Recall that a <u>path</u> in X from x_0 to x_1 is a continuous function $f : [0,1] \longrightarrow X$ s.t.

$$f(0) = x_0 \quad \text{and} \quad f(1) = x_1 \qquad \text{(we'll now keep } [0,1] \text{ as our domain)}$$

We say two paths f and g, both from x_0 to x_1, are "homotopic" if, roughly, we can continuously deform one to the other.



Here, f and g are homotopic to each other, but not to h.

i.e. we have a continuous family of paths deforming f to g.

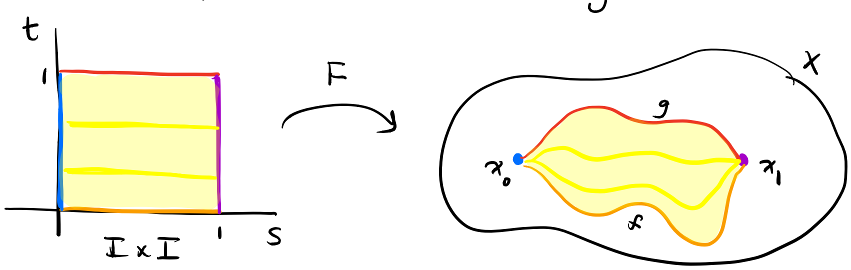
Precisely:

Def: Two paths f and g from x_0 to x_1 are <u>(path) homotopic</u> if there's a continuous map $F : \underset{[0,1]\times[0,1]}{I \times I} \longrightarrow X$ s.t.

$F(s,0) = f(s)$, $F(s,1) = g(s)$, $F(0,t) = x_0$, and $F(1,t) = x_1$

called a <u>path homotopy</u>. We write $f \simeq_p g$.

i.e. for each $t \in [0,1]$ we get another path $f_t(s) = F(s,t)$ from x_0 to x_1 "between" f and g.



Think of t as the "time" variable

We also have a more general notion of homotopy that we'll come back to later:

Def: If $f: X \to Y$ and $g: X \to Y$ are continuous, they are <u>homotopic</u> to each other if $\exists$ a continuous function $F: X \times I \to Y$ s.t. $F(x,0) = f(x)$ and $F(x,1) = g(x)$, called a <u>homotopy</u> between f and g, and we write $f \simeq g$.
If f is homotopic to a constant map, we say f is <u>nullhomotopic</u>.

Lemma: $\simeq$ and $\simeq_p$ are equivalence relations.

Pf: Clearly $f \simeq f$ s.t. $F(x,t) = f(x)$ works.

Suppose $f \simeq g$ w/ homotopy $F(x,t)$.
$G(x,t) = F(x, 1-t)$ is a (path) homotopy and gives $g \simeq f$.

Now assume $f \simeq g$ and $g \simeq h$ w/ corresponding homotopies F and G. Define $H: X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t) & \text{for } t \in [0, 1/2] \\ G(x, 2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$

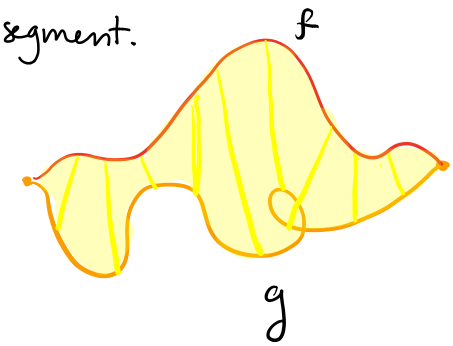
H is well-defined: if $t = \frac{1}{2}$, $F(x, 2(\frac{1}{2})) = g(x) = G(x, 2(\frac{1}{2}) - 1)$.
Thus it's continuous, and therefore a (path) homotopy. $\square$

If f is a path, we denote its path-homotopy equivalence class by $[f]$.

Ex: 1.) If f, g are paths in $\mathbb{R}^2$ from x to y, we can define the <u>straight-line homotopy</u>

$$F(s,t) = (1-t)\,f(s) + t\,g(s)$$

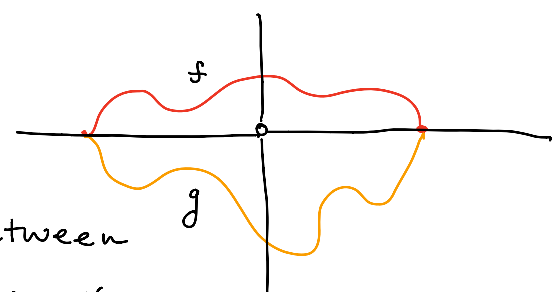
that connects the point $f(s)$ to the point $g(s)$ by a line segment.



In fact this holds for any <u>convex</u> subspace in $\mathbb{R}^n$.

2.) Let X be the "punctured plane" $\mathbb{R}^2 - \{(0,0)\}$

And f and g paths from $(-1,0)$ to $(0,1)$ s.t. the y-coordinate of f is ≥ 0, and the y-coordinate of g is ≤ 0.



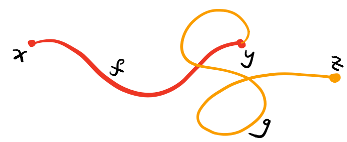
There is no homotopy between f and g in X.

We'll see precisely why later.

Products of paths

If f is a path from x to y, and g a path from y to z, We get a new path $f * g$ from x to z by running f and g twice as quickly. i.e.

$$(f * g)(s) = \begin{cases} f(2s), & s \in [0, \frac{1}{2}] \\ g(2s-1), & s \in [\frac{1}{2}, 1] \end{cases}$$



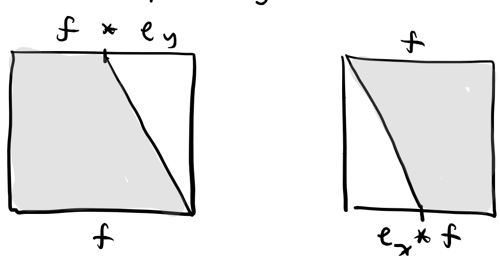
This product is well-defined on path-homotopy classes.

i.e. $[f] * [g] = [f * g]$, as long as $f(1) = g(0)$.

Does this operation have an identity? Inverses?

Identities:

Each point $x \in X$ has an identity $id_x = [e_x]$, the class of the constant path $e_x(s) = x$. Then $[f] * id_y = [f] = id_x * [f]$ by changing how quickly we run each path:



More precisely, the proof follows easily from the following claim:

Claim: If $h : X \to Y$ is continuous, and f and g are homotopic paths in X, then $h \circ f$ and $h \circ g$ are homotopic paths in Y.

$$\begin{tikzcd} I \times I \arrow[r, "F"] \arrow[dr, "\text{composition is continuous}"'] & X \arrow[d, "h"] \\ & Y \end{tikzcd}$$

Thus, if $i : [0,1] \to [0,1]$ is the identity path in $[0,1]$, then $e_0 * i \simeq i$ since $[0,1]$ is convex. But then

$f : [0,1] \to X$ is continuous, so

$$f = f \circ i \simeq f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_x * f.$$

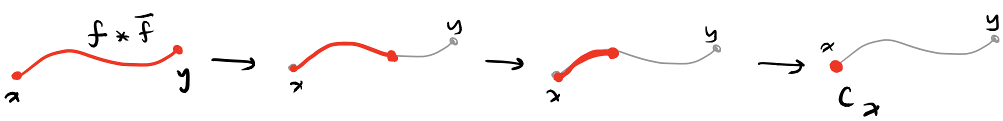
similarly, $f \simeq f * e_y$.

<u>Inverses</u>:

If f is a path from x to y, define $\bar{f}$ to be the path from y to x:

$\bar{f}(s) = f(1-s)$.

Then $f * \bar{f} \simeq_p c_x$ by the following homotopy:



We can see that this is in fact a homotopy by using the above claim and the fact that $i * \bar{i} \simeq e_0$ in $[0,1]$.

So each <u>point</u> has an "identity" path, and every path has an inverse.

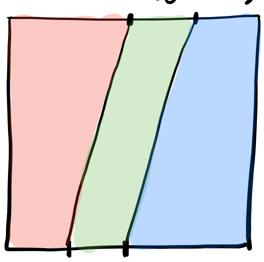
<u>Claim</u>: The path-homotopy classes form a groupoid, w/ objects the points of X, and $\mathrm{Mor}(x,y) = \{[f] : f \text{ is a path from } x \text{ to } y\}$

<u>Notice</u>: $\mathrm{Mor}(x,y)$ is nonempty $\Leftrightarrow$ x and y are in the same path component.

All that remains to show for the claim is associativity, which is trickier, but we can demonstrate using the following diagram, for f, g, h paths in X s.t.

$f(1) = g(0)$ and $g(1) = h(0)$

$f * (g * h)$



$(f * g) * h$